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Multiple phase changes induced by frustration in randomly connected cellular automata

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Abstract. Frustration is introduced in randomly connected totalistic cellular automata via mixing rules leading to incompatible periods. As the respective concentration of rules is varied, these cellular automata go through eight phases, many of which with symmetries different from the two rules mixed in. The complex phase diagram so created is similar to those seen in frustrated systems in static equilibrium. It shows that a minimized free energy is not necessary for frustration to produce this rich behaviour.

1. Introduction

The classification and understanding of temporally varying systems remain a fundamental question related to both biological and physical problems. Lacking most of the analytical and conceptual arsenal developed for equilibrium statistical mechanics, much remains to be done in this field despite considerable efforts from the community. Thus, in order to develop a good understanding of dynamical problems old concepts must be checked carefully for validity and generality and, in case of failure, new ones need to be introduced.

Of particular interest among useful concepts in equilibrium statistical physics is the question of frustration [1]. Competition between opposite forces is known to lead, in ordered systems, to very complex behaviour, as in the case of the axial next-nearest-neighbour Ising (ANNNI) model. In this model, which was proposed to explain measurements in modulated magnetic materials [2], the competition between ferromagnetic and antiferromagnetic interactions induces infinitely many ordered phases forming a so-called quasi-devil's staircase (see, for example, [3]). On the other hand, frustration can cause an important slow down in the relaxation of disordered systems, often preventing them from attaining the equilibrium state they are driven to. This effect has been studied in detail in spin glasses, see for example [4]. Although out of equilibrium, spin glasses are ultimately controlled by their drive to reach a minimum in the free energy. This concept is fundamental for the understanding of frustration in equilibrium statistical mechanics but is generally missing in dynamical systems. So, although one could expect frustration to play an important role in dynamical systems, this lack of a minimization principle could render any phenomenon engendered by such perturbation trivial.

This question has already been studied in a modified ANNNI model with detailed balance removed by the introduction of asymmetric interactions ($J_{ij} \neq J_{ji}$) [5, 6]. Although

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the location of the phases shifts slightly as a function of the level of asymmetry, the phase diagram is qualitatively preserved. The complexity under frustration existing in the equilibrium limit is thus preserved even when a system is pushed out of equilibrium. But what about dynamical systems for which an equilibrium limit does not exist?

The Chaté and Manneville cellular automata (CM-CA), which were introduced a few years ago [7], represent an appropriate toy model for addressing this question. For a subclass of the CM rules, these CA present non-trivial macroscopic behaviour such as periodic oscillation with either an integer (periodic) or irrational (quasi-periodic) frequency in units of the integer timesteps inherent to cellular automata. While displaying a macroscopic organization, these CA, started in a random configuration, continue to display stochasticity at the local level.

Hemmingsson and Peng [8] were the first to study frustration using a set of CM-CA rules. Distributing two very close rules—one with a period of three (P3) and another quasi-periodic with a period close to three (QP3)—at random on a four-dimensional hypercubic lattice they found a single phase change as they varied the proportion of each rule. This change displayed properties assimilated to a second-order phase transition: finite-size scaling, a power-law change of the order parameter and reversibility.

One can obtain a richer behaviour by mixing rules with more incompatible periods. By using a quasi-periodic rule with a period close to three (QP3) and a periodic rule with period of two (P2), I found two phase transitions as a function of the concentration of the respective rules [9]. One of them was of second order and the other one, with hysteresis, clearly first order. Besides the phases QP3 and P2, a third phase was found between the two transition lines. Without any macroscopic time structure, the CA in this central phase shows a stretched exponential decay in the autocorrelation functions, signature of a glassy phase. Dynamical systems, for which no equivalent to the free energy exists, can thus be induced by frustration into a glassy phase superimposed on their fundamental dynamics (in the case of CA, at least).

However, the question as to whether one can induce a complex phase diagram with numerous transitions in dynamical systems just by changing the amount of frustration—as is seen in the ANNNI model—remains. We know that this is possible for systems with an equilibrium limit but we cannot say anything for those where this limit does not exist. In this paper, I show that by simply varying the amount of frustration in a dynamical model one can indeed induce a large number of non-trivial phase changes. The specific phases found here are of no particular importance, but the existence of such a system may open the door to a search for more interesting objects displaying similar properties.

The study of frustration presented here is similar to the one mentioned previously in [9] except that it has been performed on a randomly connected lattice instead of a hypercubic one. A more detailed study of the general effects of random connections was presented elsewhere [10]. It was found that the macroscopic behaviour associated with the CM-CA persists even in this case. Showing a more complex behaviour on randomly connected lattices than on regular lattices, these results underline the highly non-mean-field behaviour of the CM-CA rules.

Since their introduction, the CM-CA have been extensively studied on finite-dimensional lattices but they remain imperfectly understood (see, for example, [7, 9, 11–13]). Each node, $s_i(t)$, of the CM-CA can take one of two states, 0 or 1. As they are updated synchronously, i.e. with a parallel dynamics, the state of a node at time t depends only upon the state of its neighbours and itself at time $t-1$. Moreover, in this work, the rules are fully deterministic. The update rules are totalistic with an interaction limited, in the case of random lattices, to a finite and constant set of neighbours. When the sum over the states of a site i and its neighbours falls in a bracket given by $[S_{\min}, S_{\max}]$, the state S_i is updated to 1, otherwise it is simply put to 0. This can be rewritten as follows:

$$s_i(t+1) = \begin{cases} 1 & \text{if } S_{\min} \leqslant s_i(t) + \sum_{j \in \mathcal{N}_i} s_j(t) \leqslant S_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\mathcal{N}_i$ is the neighbourhood of site i, chosen at random. Following [7], $\mathcal{R}^C_{S_{\min}-S_{\max}}$ identifies the rule used, with C the connectivity of the randomly connected lattice and $S_{\min}$ and $S_{\max}$ defined as above. The macroscopic quantity focused on is mainly the concentration of sites in state 1, called magnetization, $m(t) = 1/N \sum_i s_i(t)$, in analogy with spin systems.

Connectivity tables were prepared as in [10]. Sites are connected at random with three restrictions: (1) no two-membered rings are allowed; (2) all sites have the same number of neighbours; (3) and all bonds are taken to be symmetric. The first two restrictions are there to insure a better defined lattice, the last one is necessary in order to obtain non-trivial behaviour.

Results presented here are for a single pair of rules: $\mathcal{R}^{10}_{5-8}$ and $\mathcal{R}^{10}_{1-9}$. These rules are the same as those used in [9] on a hypercubic lattice in order to allow for some comparison. Individually, these rules do behave in a similar way to those on the five-dimensional hypercubic lattice. $\mathcal{R}^{10}_{5-8}$ shows an oscillation with a period of six (P6) but almost quasi-periodic (QP3). $\mathcal{R}^{10}_{1-9}$ on a random lattice also oscillates with a period of two but trivially, at the local level (LP2), in contrast to what happens on the hypercubic lattice. (The exact properties of the cycle for locally periodic rules depend on the initial conditions, however the qualitative behaviour remains rather constant.) Simulation results presented in this paper were obtained on a 1000 000 site lattice with a coordination of 10. Lattices were initialized at random with a macroscopic magnetization $m(0) = 0.50$. The first 500 timesteps were rejected and averages were taken over the next 1000 timesteps. Transient times in these CA are very short, typically 10–50 timesteps. The same behaviour is obtained with a smaller lattice or different initial conditions: results are fully reproducible.

Frustration is introduced here by mixing the two rules at the local level, i.e. assigning $\mathcal{R}^{10}_{5-8}$ or $\mathcal{R}^{10}_{1-9}$ as each node rule with a given probability. In equilibrium statistical physics, frustration can be understood as an introduction of competing interactions preventing local states, spins, for example, to orient themselves in such a way as to minimize all those interactions at once. A global minimization of the free energy can therefore involve the introduction of highly strained local states. Here, although no energy nor free energy function exists, we do the same: each site follows a rule favouring a given periodicity but its neighbourhood, following another rule, may push for a different periodic behaviour. Frustration here is therefore dynamical. At a given time, all sites strictly follow their rule but only the competition between different cycles appears; in some sites the periodic motion they require is not reinforced, forcing them to adopt a different cycle, while others will be able to dominate.

Figure 1 presents different phases as seen from a Poincaré map of the magnetization as a function of the concentration of rule $\mathcal{R}^{10}_{1-9}$ (p(LP2)) in the model. As the concentration of this rule is increased, the CA move through eight phases: P6, QP3, P1, P3, P1, P2, P2 $\times$ QP3, and LP2. (As will be discussed below, the cycles at p(LP2) $= 0.0$ and p(LP2) $= 0.05$ belong to the same P6 phase.) Since we still lack any theory for the behaviour of even the unfrustrated CM-CA, no satisfactory analytical description of this phenomenon is available. It is possible, however, to characterize what is seen here a little more. Since we are interested in the temporal oscillations, a natural way to study these phases is by looking at the frequency spectrum of the magnetization time series. This quantity allows a direct

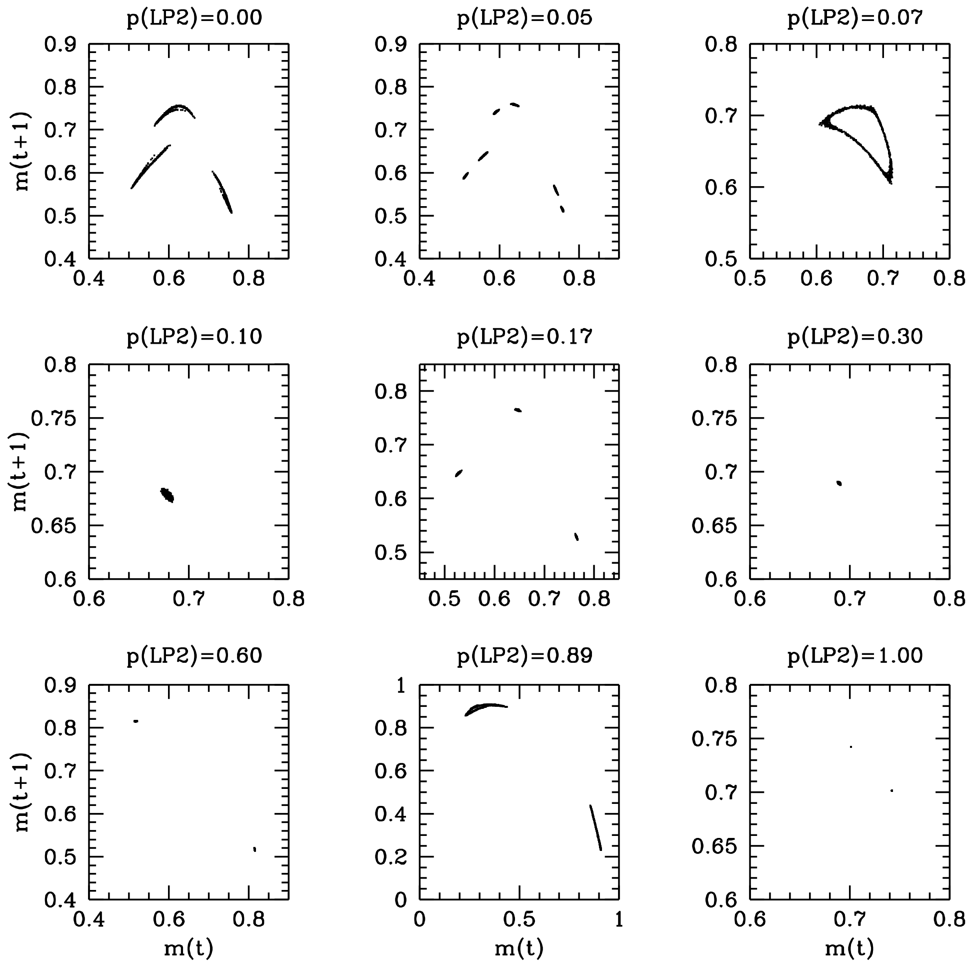

Figure 1. Poincaré maps for the macroscopic magnetization at different concentrations of rule LP2. These runs were performed on a lattice with 1000 000 sites and followed on 1000 steps after rejection of a 500-timestep transient.

measure of the symmetries associated with each phase allowing a more definite distinction between them. Figure 2 displays the frequency spectrum associated with each of the cycles of figure 1. By symmetry only frequencies up to $\frac{1}{2}$ need to be shown, and the total density $P(f)$ is normalized to 1. Here again, striking changes happen in the spectrum as the concentration p(LP2) is varied. We can also see, as mentioned earlier, that the cycles p(LP2) $=$ 0.0 and p(LP2) $=$ 0.05 possess the same symmetries (figure 3). Similarly, a more detailed comparison of the frequency spectra for p(LP2) $=$ 0.10 and p(LP2) $=$ 0.17 shows that they are qualitatively different (figure 4). We can also note the frequency spectrum of phase P2 $\times$ QP3 (p(LP2) $=$ 0.89) which clearly shows the dominant period of two as well as a contribution coming from a quasi-periodic cycle and which differs completely from the two adjacent spectra (figure 2).

All those phases are reproducible although the phase diagram is not precisely known. For example, the limits of phase P3 as well as the transition from P2 $\times$ QP3 to LP2 are not clearly defined; they fluctuate as a function of the initial conditions. All these transitions

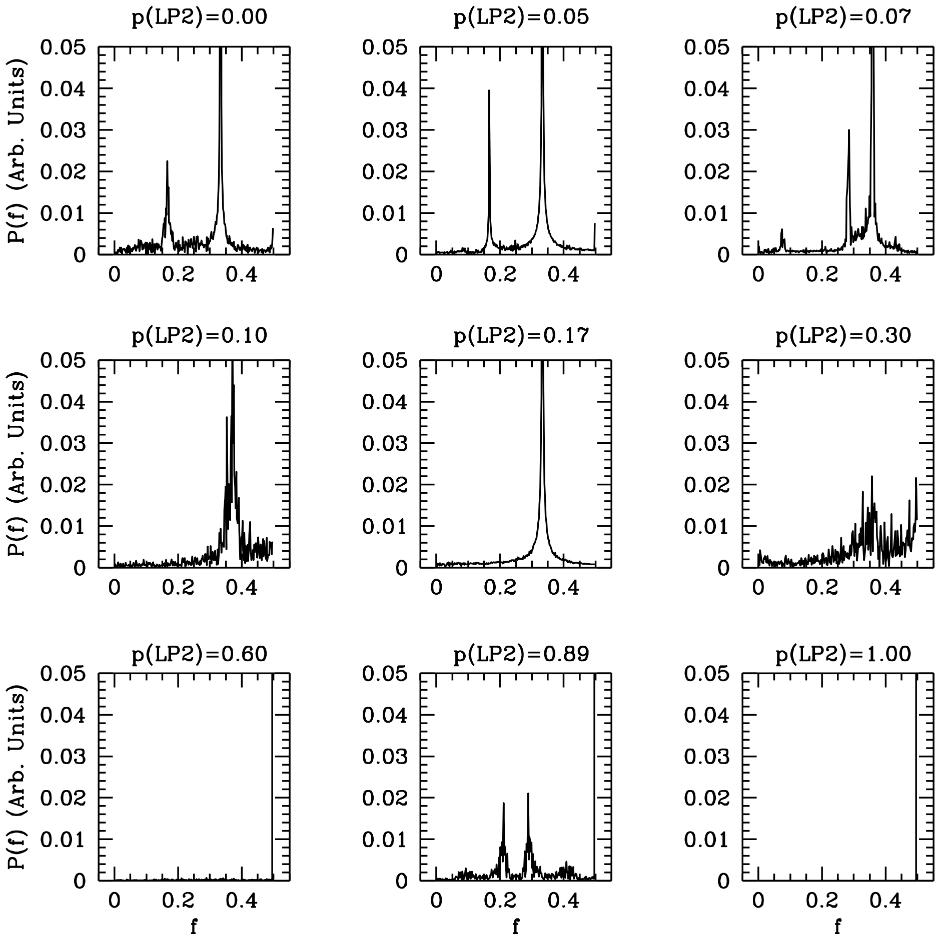

Figure 2. Frequency spectrum of magnetization time series of the previous figure. The density $P(f)$ was normalized to one and the ordinates set so as to focus on the structures. In many cases, the main peak goes up to $P(f) = 0.2$ or more.

exist nevertheless, without a doubt, as has been checked on different size lattices and multiple realizations.

It is helpful to plot a few quantities related to the macroscopic magnetization in order to get a slightly more precise idea of the phase diagram. Figure 5 plots three such quantities: the magnetization (recentred at $m = 0.5$), the fluctuations in the size of the cycle and the average frequency of the macroscopic oscillations. These three quantities show essentially the same structure and one can identify rapidly at least seven phases from the top two curves. In the average frequency plot, an eighth phase appears between p(LP2) = 0.83 and 0.89. Taking a closer look, this phase is also seen in the top two curves but only by a slight change of slope at p(LP2) = 0.83. The average frequency clearly appears more sensitive to changes than direct averages over the magnetization time series as one would expect since it connects more closely with the symmetries of the phases.

These results can be compared with previous studies of frustration discussed above which showed (a) a single phase transition [8] or (b) two phase transitions framing a glassy

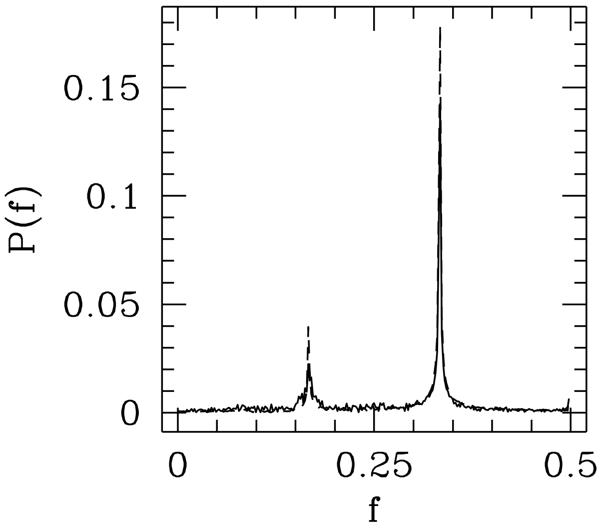

Figure 3. Comparison of the frequency spectrum for the magnetization time series for p(LP2) = 0.00 (full curve) and p(LP2) = 0.05 (broken curve). Same data as in figure 2.

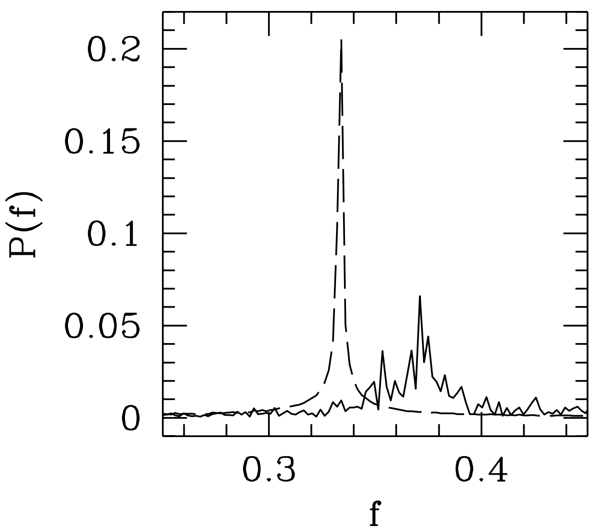

Figure 4. Comparison of the frequency spectrum for the magnetization time series for p(LP2) = 0.10 (full curve) and p(LP2) = 0.17 (broken curve). Same data as in figure 2.

region with stretched-exponential relaxation [9]. With many periods and cycles appearing, some of which displaying a richer behaviour than the pure rules, the phase diagram of the frustrated randomly connect CA is much more dramatic than these previous results. Moreover, this complexity appears in a system which should, *a priori*, be simpler than the one studied in [9] since it is described uniquely by the coordination number.

A similar study has been performed on other CM-CA rules, with different coordinations, and qualitatively similar results were found. This is therefore not a singular behaviour but it appears to be generic for these types of rules and possibly for any CA which would

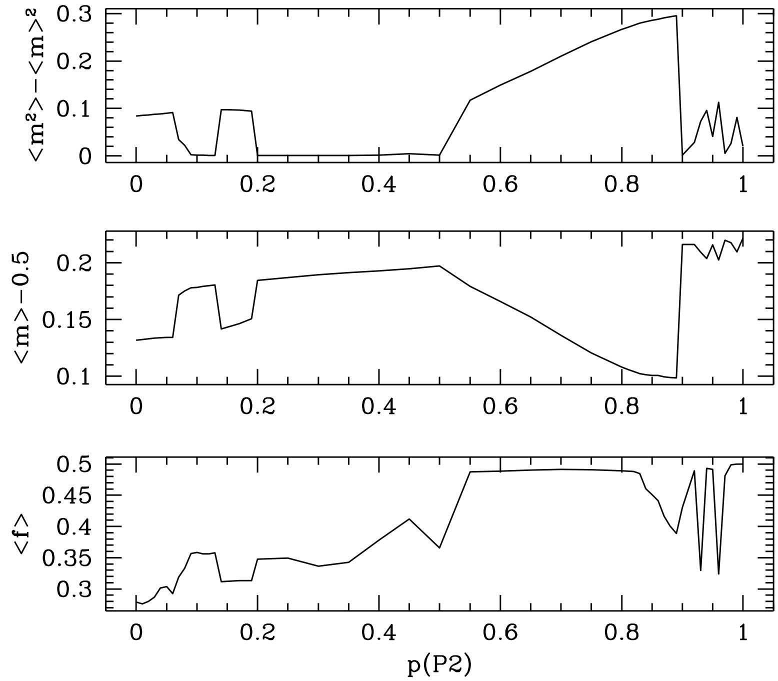

Figure 5. Variation of some averaged quantities with respect to the concentration of rule LP2 on the random lattice. Top: average fluctuations of the magnetization; middle: average magnetization; bottom: average frequencies of the Fourier spectrum, related to the size of the cycle. The structure between 0.90 and 1.0 in the lower graph is due to the local periodicity of these phases. Same parameters as in figure 1.

show similar robustness to perturbations such as the Hemmingsson rules, of which the Chaté–Manneville rules are a subset [11].

The existence of this complex phase diagram induced by a varying frustration suggests the following comments.

(1) Because of their discreteness, states of CA cannot be altered smoothly. It is therefore usually impossible to follow phases as one changes the discrete rules. By using rule mixing, one can go around the discreteness inherent to CA and follow continuously the changes in rules. Such a technique could help us to improve our understanding of some of the properties of these models.

(2) What is the role of the lattice in this problem? For rules on hypercubic lattices a single new phase, with glassy dynamics, was found. No such rich series of frustration-induced phases as for the randomly connected lattice was discovered. The more complex behaviour of a simpler lattice is rather puzzling and goes against what one would have expected. This question ties in with naturally occurring randomly connected networks such as the neurons and the general non-mean-field behaviour of these highly coordinated CA.

(3) The fact that frustration can induce new phases which are different from the ones of the pure rules in completely dynamical systems (by opposition to the results seen in the modified ANNNI case) underlines one possible source of richness in some systems around us. As previously mentioned, frustration is a very common occurrence in nature, especially in the biological context where one can often find many mechanisms competing with each

other. It could be interesting to see if some phenomena could gain from being analysed along these lines.

In conclusion, I have presented results regarding the effects of frustration in totalistic cellular automata based on a randomly connected infinite-dimensional network. As one varies the respective concentration of the different rules, one finds a series of up to eight different dynamical phases. This is much higher than what was previously found on a hypercubic lattice. It shows that frustration can act as a generator for new phases in dynamical systems, something which was known in the case of equilibrium systems, e.g. the ANNNI model, but still a matter of debate for dynamical ones. The question is now to understand how frustration can play these roles as it is no longer possible to use the familiar concept of free energy minimization to explain what happens.

Finally, this puts a new impetus on the need to find an analytical solution to the Chaté and Manneville cellular automaton in order to improve the understanding of the unusual behaviour of these frustrated models as new techniques become available [14].

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